Fourier Analysis 02-27

Review.

(Mean square convergence theorem).

Let $f \in \mathcal{R}$. Then

(a) $\lim_{N\to\infty} \| f - S_N f \| = 0$.

(b) Parseval identity

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Example 1. Let $f(x) = |x|$ on $[-\pi, \pi]$.

By a direct calculation, we have

$$f(x) \sim \frac{\pi}{2} - \sum_{n=-\infty}^{\infty} \frac{2}{\pi(2n-1)^2} e^{i(2n-1)x} \quad \text{on } [-\pi, \pi].$$

Check: $\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|^2 dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{\pi}\frac{\pi^3}{3} = \frac{\pi^2}{3}$.

By Parseval identity,

$$\frac{\pi^2}{3} = \left(\frac{\pi}{2}\right)^2 + \sum_{n=-\infty}^{\infty} \frac{4}{\pi^2} \cdot \frac{1}{(2n-1)^4} = \frac{\pi^2}{4} + \frac{8}{\pi^2} \cdot \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \left(\frac{\pi^2}{3} - \frac{\pi^2}{4}\right) \Big/ (8/\pi^2) = \frac{\pi^4}{96}.$$

Thm 2. (Riemann - Lebesgue lemma)

Let $f \in \mathcal{R}$. Then

$$\hat{f}(n) \to 0 \quad \text{as} \quad |n| \to +\infty.$$

In particular

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad \int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx$$

converge to 0 as $n \to +\infty$.

Pf.
$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \|f\|^2 < \infty$$

Hence $\hat{f}(n) \to 0$ as $|n| \to +\infty$.

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \int_{-\pi}^{\pi} f(x) \frac{e^{inx} + e^{-inx}}{2} dx$$

$$= \pi \cdot (\hat{f}(n) + \hat{f}(-n)) \to 0 \text{ as } |n| \to \infty$$

Similarly $\int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx \to 0$ as $|n| \to \infty$

$\boxtimes$.

Prop 3. Let $f, g \in \mathcal{R}$. Then

$$\langle f, g \rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n) \overline{\hat{g}(n)}.$$

Pf. It is direct to check that

$$\langle f, g \rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2 \right)$$

$$\overset{\text{by Parseval}}{=\!=} \frac{1}{4}\left[\sum_{n=-\infty}^{\infty} |\hat{f}(n)+\hat{g}(n)|^2 - |\hat{f}(n)-\hat{g}(n)|^2 + i|\hat{f}(n)-i\hat{g}(n)|^2 - i|\hat{f}(n)-i\hat{g}(n)|^2 \right]$$

$$= \sum_{n=-\infty}^{\infty} \hat{f}(n) \overline{\hat{g}(n)}.$$

$\blacksquare$

§3.4 A local convergence theorem.

Thm 4. Let $f \in \mathcal{R}$. Suppose f is differentiable at x_0. Then $\lim_{N\to\infty} S_N f(x_0) = f(x_0)$.

Pf. Recall that

$$S_N f(x_0) = f * D_N(x_0), \quad \left(\text{where } D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}.\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0-y) D_N(y)\,dy$$

$$f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0) D_N(y)\,dy$$

Hence

$$S_N f(x_0) - f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big(f(x_0-y) - f(x_0)\big)\cdot D_N(y)\,dy$$

Define
$$F(y) = \begin{cases} \dfrac{f(x_0-y)-f(x_0)}{y} & \text{if } y \in [-\pi,\pi],\ y \neq 0 \\ f'(x_0) & \text{if } y = 0. \end{cases}$$

Then F is unif. bdd on the circle, and almost all cts.
Hence $F \in \mathcal{R}$.

Then

$$\int_{-\pi}^{\pi} \left(f(x_0-y) - f(x_0) \right) D_N(y)\, dy$$

$$= \int_{-\pi}^{\pi} F(y) \cdot y \cdot D_N(y)\, dy$$

$$= \int_{-\pi}^{\pi} F(y) \cdot y \cdot \frac{\sin (N+\frac{1}{2})y}{\sin \frac{y}{2}}\, dy$$

Notice that

$$F(y) \cdot y \cdot \frac{\sin (N+\frac{1}{2})y}{\sin \frac{y}{2}} = F(y) \cdot y \cdot \frac{\sin Ny \cos \frac{y}{2} + \cos Ny \sin \frac{y}{2}}{\sin \frac{y}{2}}$$

$$= \overbrace{\left(\frac{F(y) \cdot y}{\sin \frac{y}{2}} \cdot \cos \frac{y}{2} \right)}^{:= F_1(y)} \sin Ny$$

$$+ \underbrace{\left(F(y) \cdot y \right)}_{:= F_2(y)} \cos Ny$$

Both $F_1, F_2 \in \mathcal{R}$.

Hence $$S_N f(x_0) - f(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_1(y) \sin Ny + F_2(y) \cos Ny \; dy$$

$$\to 0 \quad \text{as} \quad N \to \infty.$$

$\blacksquare$

Corollary 5. Let $f, g \in \mathcal{R}$.

Assume
$$f(x) = g(x) \quad \text{for} \quad x \in (x_0 - \varepsilon, x_0 + \varepsilon)$$
for some x_0 and $\varepsilon > 0$.

Then $\quad S_N f(x_0) - S_N g(x_0) \to 0 \quad$ as $\quad N \to +\infty$.

In particular, $S_N f(x_0)$ converges to $f(x_0)$ iff $S_N g(x_0)$ converges to $g(x_0)$.

Pf. Let $F(x) = f(x) - g(x)$.

Then $F \in \mathcal{R}$, and $F(x) = 0$ on $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$

Hence F is diff at x_0.

So by Thm 4, $\quad S_N F(x_0) \to F(x_0) = 0$ as $N \to +\infty$.

However, $S_N F(x_0) = S_N f(x_0) - S_N g(x_0)$, hence
$$S_N f(x_0) - S_N g(x_0) \to 0 \quad \text{as} \quad N \to +\infty.$$

$\boxed{///}$

The above fact was found by Riemann, it is also called the localization principle of Riemann.

It is remarkable!

Thm 6 (Dirichlet-Dini Criterion)

Let $f \in \mathcal{R}$. Suppose $\exists\, u \in \mathbb{R}$ such that

(Dini condition) $$\int_0^{\pi} \left| \frac{f(x_0+y)+f(x_0-y)}{2} - u \right| \frac{dy}{y} < \infty.$$

Then $S_N f(x_0) \to u$ as $N \to +\infty$.

Pf. Notice

$$S_N f(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0-y) \cdot D_N(y)\, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0+y)\, D_N(-y)\, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0+y)\, D_N(y)\, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x_0-y)+f(x_0+y)}{2} D_N(y)\, dy$$

$$u = \frac{1}{2\pi} \int_{-\pi}^{\pi} u\, D_N(y)\, dy$$

Hence

$$S_N f(x_0) - u = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{f(x_0-y)+f(x_0+y)}{2} - u \right) D_N(y)\, dy$$

Write $F(y) = \begin{cases} \left(\dfrac{f(x_0-y)+f(x_0+y)}{2} - u \right) \Big/ y & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$

Then $\int_0^{\pi} |F(y)|\, dy < \infty$.

Hence

$$S_N f(x_0) - u = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(y) \cdot y \cdot \frac{\sin (N+\frac{1}{2}) y}{\sin \frac{y}{2}}\, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(y) \cdot \frac{y}{\sin y/2} \cdot \cos \frac{y}{2} \sin Ny + F(y) \cdot y \cos Ny \, dy$$

$\to 0$ by Riemann-Lebesgue Lemma.

$\blacksquare$

Corollary 7. Suppose $f \in \mathcal{R}$.

Assume f is α-Hölder continuous at x_0 for some $\alpha > 0$. That is, $\exists$ a constant $C > 0$ such that

$$(*) \qquad |f(x_0+y) - f(x_0)| \leq C \cdot |y|^{\alpha} \quad \text{for all } y.$$

Then
$$\int_0^{\pi} \left| \frac{f(x_0+y) + f(x_0-y)}{2} - f(x_0) \right| \frac{dy}{y} < \infty$$

In particular, $S_N f(x_0) \to f(x_0)$ as $N \to +\infty$.

Pf. By $(*)$,
$$\left| \frac{f(x_0+y)+f(x_0-y)}{2} - f(x_0) \right| \leq C \cdot |y|^{\alpha}$$

But
$$\int_0^{\pi} C \cdot \frac{y^{\alpha}}{y} dy = C \frac{y^{\alpha}}{\alpha} \Big|_0^{\pi} = C \pi^{\alpha}/\alpha < \infty.$$

Hence
$$\int_0^{\pi} \left| \frac{f(x_0+y)+f(x_0-y)}{2} - f(x_0) \right| dy \leq \int_0^{\pi} C \cdot \frac{y^{\alpha}}{y} dy < \infty. \qquad \boxed{}$$